Int J Theor Phys (2008) 47: 2911–2923
DOI 10.1007/s10773-008-9725-y

Discrete Spectrum of the Deficit Angle and the Differential Structure of a Cosmic String

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Published online: 8 April 2008

Abstract Differential properties of Klein-Gordon and electromagnetic fields on the space-time of a straight cosmic string are studied with the help of methods of the differential space theory. It is shown that these fields are smooth in the interior of the cosmic string space-time and that they loose this property at the singular boundary except for the cosmic string space-times with the following deficit angles: $\Delta = 2\pi(1 - 1/n)$, $n = 1, 2, \ldots$.

A connection between smoothness of fields at the conical singularity and the scalar and electromagnetic conical bremsstrahlung is discussed. It is also argued that the smoothness assumption of fields at the singularity is equivalent to the Aliev and Gal'tsov "quantization" condition leading to the above mentioned discrete spectrum of the deficit angle.

Keywords Cosmic string · Differential structure · Conical bremsstrahlung · Singularities

1 Introduction

The differential structure $\mathcal{C}$ of a differential space (d-space for short) $(M, \mathcal{C})$ is a set of real functions on M which is closed with respect to localization and closed with respect to superpositions with smooth functions on $\mathbb{R}^n$, $n \in \mathbb{N}$. Every function from $\mathcal{C}$ is smooth by definition. These functions are at the base of main notions and structures defined on $(M, \mathcal{C})$. For instance, the smooth functions determine topology, tangent vectors, smooth vector fields, dimension of tangent spaces, etc. Details can be found in [9].

When one tries to describe given space-time by means of notions from the d-spaces theory the question arises. What is the meaning of smoothness and d-structure in physics of space-time? In [9, 10, 17] smooth functions from $\mathcal{C}$ are interpreted as "the system of scalar fields which actually contain all information necessary to define the manifold structure". Then the first axiom of the d-space definition, postulating the closure of $\mathcal{C}$ with respect to localization, guarantees the consistency of local physics with global physics. The second axiom, postulating the closure of $\mathcal{C}$ with respect to superposition with smooth functions

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on $\mathbb{R}^n$, provides a mechanism for construction "new" smooth quantities from "old" ones. The additional, third axiom, postulating a local diffeomorphism to $\mathbb{R}^n$, which changes a given d-space into a smooth manifold, can be interpreted as a "non-metric version of the equivalence principle" [10]. In the present paper, we test how this interpretation works in practice.

The assumptions of the present paper and the main anticipated results are the following:

1) It is supposed that space-time under investigation is a flat space-time with the conical singularity usually called the space-time of straight cosmic string.
2) Additionally, it is assumed that a scalar Klein-Gordon or electromagnetic fields are defined on the background of this space-time, and perturbations of the metric due to these fields are not taken into account.
3) The detailed analysis of differential properties of the elementary solutions for K-G scalar and electromagnetic fields (Sects. 3, 4 and 5) leads to the following results: a) The elementary solutions are smooth functions (in the sense of Sikorski) on the space-time manifold treated as a d-space. b) The above fields on this d-space with a conic singularity are smooth only for the deficit angle $\Delta = 2\pi(1 - 1/n)$, $n = 1, 2, \ldots$.

In Sect. 6 the co-called scalar and electromagnetic conical bremsstrahlung effect [2] is discussed. Section 7 contains summary of results and an argumentation that the assumption of smoothness of physical fields at the singularity (the asymptotic smoothness) can be treated as a geometric version of the Aliev and Gal'tsov condition for vanishing of the conical bremsstrahlung effect. The assumption leads to the following discrete spectrum of the deficit of angle: $\Delta = 2\pi(1 - 1/n), n = 1, 2, \ldots$. In Appendix A one can find elementary introduction to the theory of d-spaces, and in Appendix B details concerning the d-space of a cosmic string.

2 Differential Space of a Cosmic String with Singularity

Space-time described with the help of the metric

$$g = -dt^2 + k^{-2}d\rho^2 + \rho^2 d\phi^2 + dz^2 \tag{1}$$

where $k = (1 - \Delta/2\pi)$, $t, z \in \mathbb{R}$, $\rho \in (0, \infty)$ and $\phi \in \langle 0, 2\pi)$, is an example of space-time with quasiregular singularity of the conic type. The parameter $\Delta \in \langle 0, 2\pi)$ is called *deficit angle*. The three-dimensional version of the above metric is interpreted as the Schwarzchild solution in the framework of 3-D gravity [28], whereas the four-dimensional metric is interpreted as an exterior gravitational field of a straight cosmic string in 4-D gravity [29].

The space-time of a cosmic string as a pseudoriemannian manifold (M, g) is isometric to $(C^\circ \times \mathbb{R}^2, \iota^*\eta^{(5)})$, where C° is a two-dimensional cone without the vertex; $\iota : C^\circ \times \mathbb{R}^2 \to \mathbb{R}^5$ is an embedding and $\eta^{(5)}$ is the five-dimensional Minkowski metric. The space-time of a cosmic string as a differential space is diffeomorphic to $C^\circ \times \mathbb{R}^2$, where the latter is treated as a differential subspace of the d-space $(\mathbb{R}^5, \mathcal{E}_5)$, where $\mathcal{E}_5 = C^\infty(\mathbb{R}^5)$. In other words, (M, g) as a d-space $(M, \mathcal{M})$ is diffeomorphic to $(C^\circ \times \mathbb{R}^2, (\mathcal{E}_5)_{C^\circ \times \mathbb{R}^2})$, where $(\mathcal{E}_5)_{C^\circ \times \mathbb{R}^2}$ is the induced d-structure [8, 9] and the symbol $(\cdot)_{C^\circ \times \mathbb{R}^2}$ denotes the operation of taking closure with respect to localization (Definition A.2).

The singular space-time of a cosmic string can be defined in various ways. The most popular method depends on attaching the vertex of the cone to C°. Then $C^\bullet \times \mathbb{R}^2$ represents the space-time of the cosmic string with singularity, where $C^\bullet$ denotes the cone with the vertex. Evidently, $C^\bullet \times \mathbb{R}^2$ is not a sub-manifold of $\mathbb{R}^5$ but it is still a d-subspace: $(C^\bullet \times \mathbb{R}^2, (\mathcal{E}_5)_{C^\bullet \times \mathbb{R}^2})$.

However, the main objects of our study in the present paper are two auxiliary d-spaces $(P^\circ, \mathcal{P}^\circ)$ and $(P^\bullet, \mathcal{P}^\bullet)$ (for details see Appendix B or [8]) which are diffeomorphic to the d-space of cosmic string without singularity and to the d-space of cosmic string with singularity, respectively. The auxiliary d-spaces are more convenient for investigations than the original ones $(C^\circ \times \mathbb{R}^2, (\mathcal{E}_5)_{C^\circ \times \mathbb{R}^2})$ and $(C^\bullet \times \mathbb{R}^2, (\mathcal{E}_5)_{C^\bullet \times \mathbb{R}^2})$. For example, the above described singularity attaching process is simply the procedure of taking limits of a few functions from $\mathcal{P}^\circ$ [8]. The resulting d-space $(P^\bullet, \mathcal{P}^\bullet)$ is, in a sense, a limit (or an asymptotic) state of the background d-space $(P^\circ, \mathcal{P}^\circ)$.

3 Differential Properties of a Scalar Field in the Conical Space-Time of a Cosmic String

As well known, normal modes of the Klein-Gordon scalar field for the cosmic string space-time in (t, ρ, ϕ, z) coordinates have the following form

$$\begin{gathered} \tilde{\Psi}^\circ_{\epsilon,l,\beta} \colon \tilde{P}^\circ \to \mathbb{C}, \\ \tilde{\Psi}^\circ_{\epsilon,l,\beta}(t, \rho, \phi, z) = N_{\epsilon,l,\beta} e^{-i\epsilon t} e^{i\beta z} e^{il\phi} \rho^{|l|/k} F(l, k; \rho) \end{gathered} \tag{2}$$

where $\epsilon, \beta \in \mathbb{R}$, $l \in \mathbb{Z}$, $N_{\epsilon,l,\beta}$ is the normalization constant, $k \in (0, 1)$ is defined in formula (1) and $\tilde{P}^\circ$ is given in Appendix B. $F(l, k; \rho)$ is an analytical function of ρ. Its detailed form is not important for the present study.

Additionally, there is an another set of normal modes usually excluded from physical investigations because of a singular behaviour of functions as their arguments tend to the singularity ($\rho \to 0$) [24]. Sometimes however, such normal modes can be of physical relevance [12] but in the present paper this divergent normal modes are not taken into account.

The space-time of the cosmic string is represented by the d-space $(P^\circ, \mathcal{P}^\circ)$. Therefore, in this case normal modes of the scalar field are given by:

$$\begin{gathered} \Psi^\circ_{\epsilon,l,\beta} \colon P^\circ \to \mathbb{C}, \\ \Psi^\circ_{\epsilon,l,\beta}([p]) := \tilde{\Psi}^\circ_{\epsilon,l,\beta}(p), \end{gathered} \tag{3}$$

where $p = (t, \rho, \phi, z)$, $[p] \in P^\circ = \tilde{P}^\circ/\rho_H$ (Appendix B).

Definition 3.1 Let $(M, \mathcal{C})$ be a d-space. A complex function $f \colon M \to \mathbb{C}$ is said to be smooth if $\mathrm{Re} f, \mathrm{Im} f \in \mathcal{C}$.

Proposition 3.1 *The normal modes $\Psi^\circ_{\epsilon,l,\beta}$ are smooth functions on $(P^\circ, \mathcal{P}^\circ)$ for every $\epsilon, \beta \in \mathbb{R}$, $l \in \mathbb{Z}$ and $k \in (0, 1)$.*

Proof Following Corollary A.1 it is enough to prove the smoothness of $\tilde{\Psi}^\circ_{\epsilon,l,\beta}$ on $(\tilde{P}^\circ, \tilde{\mathcal{P}}^\circ)$ (Appendix B). It is easy to check that functions $\tilde{\Psi}^\circ_{\epsilon,l,\beta}$ are smooth functions since they are superpositions of generators $\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_4$ (see Appendix B) with smooth functions from $C^\infty(\mathbb{R}^m)$, $m = 1, 2, \ldots$ and $\rho^{|l|/k}$ is smooth for $\rho > 0$. □

According to Proposition 3.1 the normal modes (3) are smooth functions on $(P^\circ, \mathcal{P}^\circ)$. They belong to $\mathcal{P}^\circ$ and therefore they carry no new information from the point of view of the "pre-geometry" determined by $\mathcal{P}^\circ$. One can say that the space-time of a cosmic string is "prepared" for imposing a scalar field on its background d-space. In other words, space-time is from the very beginning differentially configured in such a manner that the imposition of a scalar field is done by the indication which functions among already existing ones in $\mathcal{P}^\circ$ are normal modes. This fact is a confirmation of the correctness of the assumption that the scalar field on a given space-time does not cause any changes of properties of the gravitational field.

$(P^\circ, \mathcal{P}^\circ)$ represents the background d-space of a cosmic string if $(P^\bullet, \mathcal{P}^\bullet)$ is its asymptotic state. This statement mirrors the fact that the gravitational field described by means of metric (1) is a space-time of a cosmic string only if a singular boundary of a conic type is present. Therefore, in the process of calculating normal modes, its asymptotic properties at the singularity have to be taken into account. This kind of analysis excludes divergent modes from further field-theory considerations [24]. The question arises: What are differential properties of the normal modes at the singularity? The following argumentation clarifies the situation.

One can easily check that normal modes naturally prolonged to singularity,

$$\tilde{\Psi}^\bullet_{\epsilon,l,\beta}(p) := \lim_{q\to p} \tilde{\Psi}^\circ_{\epsilon,l,\beta}(q), \quad q \in \tilde{P}^\circ, p \in \tilde{P}^\bullet, \tag{4}$$

are constant functions on every equivalence class $[p]$ for $p \in \tilde{P}^\bullet$ (see Appendix A, formula (7) and Appendix B). Therefore, they can be used for construction prolonged modes $\Psi^\bullet_{\epsilon,l,\beta}$ defined on $(P^\bullet, \mathcal{P}^\bullet)$:

$$\Psi^\bullet_{\epsilon,l,\beta}([p]) := \tilde{\Psi}^\bullet_{\epsilon,l,\beta}(p), \quad [p] \in P^\bullet, p \in \tilde{P}^\bullet. \tag{5}$$

The prolongation is natural from the physical point of view.

Proposition 3.2 *For every $\epsilon, \beta \in \mathbb{R}$ and $l \in \mathbb{Z}$, $\Psi^\bullet_{\epsilon,l,\beta}$ are*

a) *smooth functions on $(P^\bullet, \mathcal{P}^\bullet)$ for $k \in (0, 1)$ such that $|l|/k \in \mathbb{N}$,*
b) *non-smooth functions on $(P^\bullet, \mathcal{P}^\bullet)$ for $k \in (0, 1)$ such that $|l|/k \notin \mathbb{N}$.*

Proof It is enough to check smoothness for $\tilde{\Psi}^\bullet_{\epsilon,l,\beta}$. Functions $e^{-i\epsilon t}$, $e^{i\beta z}$, $e^{il\phi}$, $F(l, k; \rho)$ are smooth owing to the same arguments as in Proposition 3.1. The function $\rho^{|l|/k} = (\alpha_4(\rho))^{|l|/k}$ is a smooth function for $\rho > 0$. At $\rho = 0$ the superposition is smooth the only in the case $|l|/k \in \mathbb{N}$. □

In general, $\Psi^\bullet_{\epsilon,l,\beta}$ are not smooth functions on $(P^\bullet, \mathcal{P}^\bullet)$except for cases when the metric parameter $k \in (0, 1)$ satisfies the condition: $\forall\, l \in \mathbb{Z} : |l|/k \in \mathbb{N}$. This means that $k = 1/n$, where $n = 2, 3, \ldots$. One can also include the case of the Minkowski space ($k = 1$). In other words, the prolonged normal modes are smooth functions only for the space-time of a cosmic string with the deficit angle $\Delta = 2\pi(1 - 1/n)$, where $n = 1, 2, \ldots$.

4 Global Properties of the Space-Time of a Cosmic String with a Scalar Field

Let a real-valued function $\beta^\bullet : P^\bullet \to \mathbb{R}$ be defined by the formula

$$\beta^\bullet([p]) := \tilde{\beta}^\bullet(p) := \rho^{1/k}, \tag{6}$$

where $k \in (0,1\rangle$. One can define an another d-structure on $P^\bullet$:

$$\hat{\mathcal{P}}^\bullet = \mathrm{Gen}(\alpha_0^\bullet, \alpha_1^\bullet, \ldots, \alpha_4^\bullet, \beta^\bullet).$$

Proposition 4.1 *For $k \neq 1/n$, $n = 1, 2, 3, \ldots$, the prolonged normal modes $\Psi^\bullet_{\epsilon,l,\beta}$ are smooth functions on $(P^\bullet, \hat{\mathcal{P}}^\bullet)$ for every $\epsilon, \beta \in \mathbb{R}$ and $l \in \mathbb{Z}$.*

Proof The factor $\rho^{|l|/k}$ in formula (2), after prolongation to the singular boundary, is a smooth composition of $\beta^\bullet$. □

Corollary 4.1 *If $k = 1/n$, $n = 1, 2, 3, \ldots$ then $(P^\bullet, \hat{\mathcal{P}}^\bullet) = (P^\bullet, \mathcal{P}^\bullet)$.*

Proof $\beta^\bullet \in \mathcal{P}^\bullet$ for $k = 1/n$. □

Proposition 4.2 *If $k \neq 1/n$, $n = 1, 2, 3, \ldots$ then the set $\{\alpha_0^\bullet, \alpha_1^\bullet, \ldots, \alpha_4^\bullet, \beta^\bullet\}$ is differentially independent at boundary points $p \in \mathbf{S} = P^\bullet - P^\circ$. $\beta^\bullet$ and $\alpha_4^\bullet$ differentially depend on $\{\alpha_0^\bullet, \alpha_1^\bullet, \alpha_2^\bullet, \alpha_3^\bullet\}$ elsewhere on $P^\bullet$.*

Proof The conclusion is a straightforward consequence of Definitions A.6 and A.7. □

Corollary 4.2 *If $k \neq 1/n$ then*

a) $\dim T_p(P^\bullet, \hat{\mathcal{P}}^\bullet) = 6$ *for* $p \in \mathbf{S}$,
b) $\dim T_p(P^\bullet, \hat{\mathcal{P}}^\bullet) = 4$ *for* $p \notin \mathbf{S}$,

where $\mathbf{S} = P^\bullet - P^\circ$ denotes the set of all boundary points.

Proof The conclusion is a consequence of Lemma A.1 and Proposition 4.2 □

In general, $\Psi^\bullet_{\epsilon,l,\beta}$ are non-smooth functions on $(P^\bullet, \mathcal{P}^\bullet)$because of the $\rho^{|l|/k}$ factor in formula (2). Other factors are smooth functions. In order to keep smoothness of $\Psi^\bullet_{\epsilon,l,\beta}$ on $P^\bullet$ one has to modify the d-structure $\mathcal{P}^\bullet$ by adding "necessary" functions. In our case, the most physically reasonable method is to supplement the set of generators $\{\alpha_0^\bullet, \alpha_1^\bullet, \ldots, \alpha_4^\bullet\}$ of the d-structure $\mathcal{P}^\bullet$ (Appendix B) with the function $\beta^\bullet$. Then $\hat{\mathcal{P}}^\bullet = \mathrm{Gen}(\alpha_0^\bullet, \alpha_1^\bullet, \ldots, \alpha_4^\bullet, \beta^\bullet)$ is the smallest d-structure containing $\alpha_0^\bullet, \alpha_1^\bullet, \ldots, \alpha_4^\bullet$ and $\beta^\bullet$ as smooth functions [8, 9].

Thus, in order to keep smoothness of $\Psi^\bullet_{\epsilon,l,\beta}$, the prolonged d-space of the cosmic string space-time with a scalar field ought to be represented by $(P^\bullet, \hat{\mathcal{P}}^\bullet)$ rather than by $(P^\bullet, \mathcal{P}^\bullet)$. In the case $k = 1/n$, the function $\beta^\bullet$ is smooth on $(P^\bullet, \mathcal{P}^\bullet)$ and according to Corollary 4.1 the prolonged background d-spaces both for the cosmic string space-time and the cosmic string space-time with a scalar field are the same; $(P^\bullet, \hat{\mathcal{P}}^\bullet) = (P^\bullet, \mathcal{P}^\bullet)$. In the case $k \neq 1/n$, $\beta^\bullet$ is not a smooth function on $(P^\bullet, \mathcal{P}^\bullet)$. This means that $(P^\bullet, \hat{\mathcal{P}}^\bullet)$ and $(P^\bullet, \mathcal{P}^\bullet)$ are different and are not diffeomorphic d-spaces. For example $(P^\bullet, \hat{\mathcal{P}}^\bullet)$ has differential dimension 6 at the singular points (Corollary 4.2) whereas the $(P^\bullet, \mathcal{P}^\bullet)$—5 (see [8]). A straightforward consequence of this fact is that $(P^\bullet, \hat{\mathcal{P}}^\bullet)$ cannot be embedded in $\mathbb{R}^5$ like $(P^\bullet, \mathcal{P}^\bullet)$, but can be embedded in $\mathbb{R}^6$.

In order to formulate the final conclusions of the above discussion let us define the following.

Definition 4.1 A physical scalar (vector, tensor) field Φ is said to be smooth on a background d-space $(M, \mathcal{C})$ iff scalar (vector, tensor) elementary solutions of the corresponding field equation are smooth functions (vector, tensor fields) on $(M, \mathcal{C})$.

Definition 4.2 A physical scalar (vector, tensor) field Φ is said to be asymptotically smooth on $(M^\circ, \mathcal{C}^\circ, M^\bullet, \mathcal{C}^\bullet, g)$ if

a) Φ is a smooth physical field on $(M^\circ, \mathcal{C}^\circ)$,
b) Φ is a smooth physical field on $(M^\bullet, \mathcal{C}^\bullet)$,

where the symbol $(M^\circ, \mathcal{C}^\circ, M^\bullet, \mathcal{C}^\bullet, g)$ denotes a pseudoriemannian manifold (M°, g) equipped with a metric g which, as a d-space $(M^\circ, \mathcal{C}^\circ)$, has the prolongation $(M^\bullet, \mathcal{C}^\bullet)$.

The prolonged background d-space of a cosmic string is represented strictly by $(P^\bullet, \mathcal{P}^\bullet)$ (see Appendix B) so $(P^\bullet, \hat{\mathcal{P}}^\bullet)$ cannot be interpreted in such manner. Thus, including Minkowski space-time ($k = 1$), one can formulate the following theorem.

Theorem 4.1 *The conical space-time of a cosmic string* $(P^\circ, \mathcal{P}^\circ, P^\bullet, \mathcal{P}^\bullet, g)$ *can be a background of an asymptotically smooth Klein-Gordon scalar field only in the case of the following discrete spectrum of the deficit angle*:

$$\Delta = 2\pi(1 - 1/n),$$

where $n = 1, 2, \ldots,$ *and* g *denotes metric* (1).

I would like to emphasize that the discrete spectrum of the deficit angle is not visible on the "metric" level. The effect appears when the differential properties of a cosmic string are taken into account.

5 Electromagnetic Field in a Conical Space-Time

Let $\tilde{A}^\circ{}_\sigma\colon \tilde{P}^\circ \to \mathbb{C}$ be an electromagnetic field. $\tilde{A}^\circ{}_\sigma$, in the Lorentzian gauge $\nabla^\mu \tilde{A}^\circ{}_\mu = 0$, obey the Maxwell equations $\nabla_\mu \nabla^\mu \tilde{A}^\circ{}_\sigma = 0$. Their elementary solutions have the following form

$$\begin{aligned}
\tilde{A}^\circ{}_a(\epsilon, \beta, l) &= e^{-i\epsilon t} e^{i\beta z} e^{il\phi} \rho^{|l|/k} F^a_{\epsilon,\beta,l,k}(\rho),\\
\tilde{A}^\circ{}_1(\epsilon, \beta, l) &= e^{-i\epsilon t} e^{i\beta z} e^{il\phi} \left(\rho^{|1+l/k|} F^1_{\epsilon,\beta,l,k}(\rho) + \rho^{|1-l/k|} F^2_{\epsilon,\beta,l,k}(\rho)\right),\\
\tilde{A}^\circ{}_2(\epsilon, \beta, l) &= \frac{k\rho}{2i} e^{-i\epsilon t} e^{i\beta z} e^{il\phi} \left(\rho^{|1+l/k|} F^1_{\epsilon,\beta,l,k}(\rho) - \rho^{|1-l/k|} F^2_{\epsilon,\beta,l,k}(\rho)\right),
\end{aligned}$$

where $\epsilon, \beta \in \mathbb{R}$, $l \in \mathbb{Z}$ and $a = 0, 3$. Detailed forms of the analytical functions $F^b_{\epsilon,\beta,l,k}$, $b = 0, 1, 2, 3$ can be found in [2] but they are not relevant for further discussion.

Definition 5.1 Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two sets of real functions on M and $\mathbf{V}_m\colon \mathcal{C} \to \mathcal{B}_m$, $\mathcal{B}_m = \mathbf{V}_m(\mathcal{C})$, $m = 1, 2$, be vector fields on $(M, \mathcal{C})$. The mapping $\mathbf{V} := \mathbf{V}_1 + i\mathbf{V}_2$ is said to be a complex vector field on $(M, \mathcal{C})$. $\mathbf{V}$ is a smooth complex vector field on $(M, \mathcal{C})$ if $\mathcal{B}_m \subset \mathcal{C}$ for $m = 1, 2$.

Let us define complex vector fields $\tilde{\mathbf{A}}^{\circ}{}_{\epsilon,\beta,l}$ and $\tilde{\mathbf{A}}^{\bullet}{}_{\epsilon,\beta,l}$ on $(\tilde{P}^{\circ},\tilde{\mathcal{P}}^{\circ})$ and $(\tilde{P}^{\bullet},\tilde{\mathcal{P}}^{\bullet})$, respectively, by the following formulae

$$\tilde{\mathbf{A}}^{\circ}{}_{\epsilon,\beta,l} := \tilde{A}^{\circ\mu}\partial_{\mu},$$

$$\tilde{\mathbf{A}}^{\bullet}{}_{\epsilon,\beta,l}(\gamma^{\bullet})(p) := \lim_{q\to p}\tilde{\mathbf{A}}^{\circ}{}_{\epsilon,\beta,l}(\gamma^{\circ})(q),$$

where $\gamma^{\circ} := \gamma^{\bullet}\mid_{\tilde{P}^{\circ}}$, $\gamma^{\bullet}\in\tilde{\mathcal{P}}^{\bullet}$, $q\in\tilde{P}^{\circ}$ and $p\in\tilde{P}^{\bullet}$.

Then, electromagnetic vector fields $\mathbf{A}^{\circ}{}_{\epsilon,\beta,l}$ and $\mathbf{A}^{\bullet}{}_{\epsilon,\beta,l}$ on $(P^{\circ},\mathcal{P}^{\circ})$ and $(P^{\bullet},\mathcal{P}^{\bullet})$, respectively, can be defined in the following way

$$\mathbf{A}^{\circ}{}_{\epsilon,\beta,l} := \pi^{\#}_{\rho_H}(\tilde{\mathbf{A}}^{\circ}{}_{\epsilon,\beta,l}),$$

$$\mathbf{A}^{\bullet}{}_{\epsilon,\beta,l} := \pi^{\#}_{\rho_H}(\tilde{\mathbf{A}}^{\bullet}{}_{\epsilon,\beta,l}),$$

where the map $\pi^{\#}_{\rho_H}$ is given in Appendix A.

Proposition 5.1 *For every $\epsilon,\beta\in\mathbb{R}$ and $l\in\mathbb{Z}$*

1. $\mathbf{A}^{\circ}{}_{\epsilon,\beta,l}$ *are smooth complex vector fields on* $(P^{\circ},\mathcal{P}^{\circ})$,
2. $\mathbf{A}^{\bullet}{}_{\epsilon,\beta,l}$ *are*
   a) *smooth complex vector fields on* $(P^{\bullet},\mathcal{P}^{\bullet})$, *for* $k\in(0,1)$, *such that* $|l|/k\in\mathbb{N}$,
   b) *not smooth complex vector fields on* $(P^{\bullet},\mathcal{P}^{\bullet})$, *for* $k\in(0,1)$, *such that* $|l|/k\notin\mathbb{N}$.

Proof Following Corollary A.2, $\mathbf{A}^{\circ}{}_{\epsilon,\beta,l}$ and $\mathbf{A}^{\bullet}{}_{\epsilon,\beta,l}$ are smooth vector fields on $(P^{\circ},\mathcal{P}^{\circ})$ and $(P^{\bullet},\mathcal{P}^{\bullet})$, respectively, iff both $\tilde{\mathbf{A}}^{\circ}{}_{\epsilon,\beta,l}$ on $(\tilde{P}^{\circ},\tilde{\mathcal{P}}^{\circ})$ and $\tilde{\mathbf{A}}^{\bullet}{}_{\epsilon,\beta,l}$ on $(\tilde{P}^{\bullet},\tilde{\mathcal{P}}^{\bullet})$ are smooth. Straightforward calculations lead to the conclusion that $\mathrm{Re}\,\tilde{\mathbf{A}}^{\circ}{}_{\epsilon,\beta,l}(\tilde{\mathcal{P}}^{\circ})\subset\tilde{\mathcal{P}}^{\circ}$, $\mathrm{Im}\,\tilde{\mathbf{A}}^{\circ}{}_{\epsilon,\beta,l}(\tilde{\mathcal{P}}^{\circ})\subset\tilde{\mathcal{P}}^{\circ}$, for every $k\in(0,1)$, and $\mathrm{Re}\,\tilde{\mathbf{A}}^{\bullet}{}_{\epsilon,\beta,l}(\tilde{\mathcal{P}}^{\bullet})\subset\tilde{\mathcal{P}}^{\bullet}$, $\mathrm{Im}\,\tilde{\mathbf{A}}^{\bullet}{}_{\epsilon,\beta,l}(\tilde{\mathcal{P}}^{\bullet})\subset\tilde{\mathcal{P}}^{\bullet}$ only for $k\in(0,1)$ such that $|l|/k\in\mathbb{N}$ for every $l\in\mathbb{Z}$. $\square$

Theorem 5.1 *The conical space-time of a cosmic string $(P^{\circ},\mathcal{P}^{\circ},P^{\bullet},\mathcal{P}^{\bullet},g)$ can be a background of an asymptotically smooth electromagnetic field only in the case of the following discrete spectrum of the deficit angle*:

$$\Delta = 2\pi(1-1/n),$$

where $n=1,2,\ldots$, and g denotes metric (1).

Proof The argumentation is similar to that of in the scalar field case. Following Proposition 5.1 and Definition 4.2, $\mathbf{A}^{\bullet}{}_{\epsilon,\beta,l}$ are not smooth complex vector fields on $(P^{\bullet},\mathcal{P}^{\bullet})$ for $k\neq 1/n$, $n=1,2,\ldots$ and therefore they are not asymptotically smooth on $(P^{\circ},\mathcal{P}^{\circ},P^{\bullet},\mathcal{P}^{\bullet},g)$. For $k=1/n$, $n=1,2,\ldots$ both $\mathbf{A}^{\circ}{}_{\epsilon,\beta,l}$ and $\mathbf{A}^{\bullet}{}_{\epsilon,\beta,l}$ are smooth complex vector fields on $(P^{\circ},\mathcal{P}^{\circ})$ and $(P^{\bullet},\mathcal{P}^{\bullet})$ respectively. $\square$

6 Radiative Aharonov-Bohm Effect and Differential Structures

The space-time of a cosmic string is locally flat and consequently there are no local gravitational forces acting on massive bodies or light rays. In spite of this, there are a few interesting effects such as: the lensing effect [6, 30], production of an electromagnetic radiation by a freely moving charge [20], radiative "conical bremsstrahlung" [1, 2], which are examples of the gravitational Aharonov-Bohm effects [5, 30]. From the point of view of the present paper the most interesting are the so-called radiative A-B effects appearing when a scalar or charged particle is moving in the space-time of a cosmic string [2].

A scalar (electric) charge freely moving in the space-time of a cosmic string can be regarded as a source of a scalar (electromagnetic) field with non-vanishing energy-momentum tensor. During the motion a variation of the total energy of the field appears. The variation $\mathcal{E}$ is interpreted by Aliev and Gal'tsov as the total work done by a radiation friction force upon the source. The effect is called *scalar* (*electromagnetic*) *conical bremsstrahlung* [2].

In the scalar and electromagnetic cases, the distribution of variation of the total energy is of the form

$$\frac{d\mathcal{E}}{d\omega} = \frac{\sin^2(\pi/k)}{\pi k} F_{sc(em)}(k, q, \omega, d, v, U^0),$$

where $F_{sc(em)}$ is a function of k, ω and constants of motion. Its detailed form can be found in the original paper by Aliev and Gal'tsov [2]. However, for our purposes only the dependence on k is important. In both scalar and electromagnetic cases, $\mathcal{E}$ vanishes for $k = 1/n$ $(\Delta = 2\pi(1 - 1/n))$, $n = 1, 2, \ldots$.

Thus, there is an apparent connection between the smoothness of the elementary solutions on the d-space of a cosmic string with singularity $(P^\bullet, \mathcal{P}^\bullet)$ and the effect of vanishing of $\mathcal{E}$. The nature of this connection has a relatively simple mathematical origin. Namely, the variation of the total energy $\mathcal{E}$ is calculated by means of so-called radiative Green function which is constructed with the help of the elementary solutions $\tilde{\Psi}^\bullet_{\epsilon,l,\beta}$ in the scalar case and with the help of $\tilde{A}^\bullet_\mu(\epsilon, \beta, l)$ in the electromagnetic case. As shown in [1, 2, 20], $\mathcal{E}$ constructed in such a manner vanishes for smooth elementary solutions on $(P^\bullet, \mathcal{P}^\bullet)$, and is different from zero for non-smooth ones.

7 Summary and Discussion

The main purpose of the present paper is to test whether, and in what way, physical fields on the space-time of a cosmic string participate in the formation of the manifold structure $\mathcal{P}^\circ$ and the d-structure $\mathcal{P}^\bullet$, where $\mathcal{P}^\bullet$ represent the d-structure of the space-time of a cosmic string with the singular boundary (see Sect. 1 and [7, 10]).

Mathematically, the test is based on verifying whether the elementary solutions of a scalar field belong to $\mathcal{P}^\circ$ or, after prolongation to $\mathcal{P}^\bullet$. In the case of an electromagnetic field, one tests the smoothness of $\mathbf{A}^\circ{}_{\epsilon,\beta,l}$ and $\mathbf{A}^\bullet{}_{\epsilon,\beta,l}$ on $(P^\circ, \mathcal{P}^\circ)$ and $(P^\bullet, \mathcal{P}^\bullet)$, respectively.

Propositions 3.1 and 5.1.1 state that, indeed, the physical fields in the interior of the cosmic string space-time P° participate in the formation of the manifold structure $\mathcal{P}^\circ$ in such a way that they can be reconstructed by means of the original space-time generators $\{\alpha_0^\circ, \alpha_1^\circ, \ldots, \alpha_4^\circ\}$ by using the operation of taking closure with respect to superposition with smooth functions on $\mathbb{R}^n$ (see Appendix A). This is consistent with the interpretation mentioned in Sect. 1 or in [10].

A new situation appears when one takes into consideration the cosmic string space-time with singularity. Such an object is not a manifold, but it is still a d-space $(P^\bullet, \mathcal{P}^\bullet)$. With the exception of the cosmic string space-times with $\Delta = 2\pi(1 - 1/n)$, $n = 1, 2, \ldots$, scalar and electromagnetic fields do not participate in the formation of the original d-structure $\mathcal{P}^\bullet$ (Propositions 3.2 and 5.1.2).

Thus, if one assumes that space-time of a cosmic string is a pseudoriemannian manifold (P°, g) which, as the d-space $(P^\circ, \mathcal{P}^\circ)$, has the prolongation $(P^\bullet, \mathcal{P}^\bullet)$ (Sect. 2 and Definition 4.2) then it can be a background of an asymptotically smooth scalar field or of an asymptotically smooth electromagnetic field only for the following deficit angles $\Delta = 2\pi(1-1/n)$, $n = 1, 2, \ldots$ (Theorems 4.1 and 5.1).

However, it is also interesting to test what happens if the elementary solutions are assumed to be smooth functions on the whole of $P^\bullet$ even in the case $\Delta \neq 2\pi(1-1/n)$, $n = 1, 2, \ldots$. In the present paper the consequences of such an assumption were discussed for the case of a scalar K-G field (Sect. 4). It turns out that the assumption of smoothness of the scalar elementary solutions (normal modes) is satisfied when the d-structure on $P^\bullet$ is $\hat{\mathcal{P}}^\bullet$ instead of $\mathcal{P}^\bullet$. $\hat{\mathcal{P}}^\bullet$ is the smallest d-structure (in the sense of inclusion) which contains the scalar elementary solutions. The space-time with singularity $P^\bullet$ and the d-structure $\hat{\mathcal{P}}^\bullet$ forms a d-space $(P^\bullet, \hat{\mathcal{P}}^\bullet)$ which is not diffeomorphic to the original background d-space with singularity $(P^\bullet, \mathcal{P}^\bullet)$. For example, $(P^\bullet, \hat{\mathcal{P}}^\bullet)$ differs from $(P^\bullet, \mathcal{P}^\bullet)$ by its embedding properties. It cannot be embedded in $\mathbb{R}^5$ like $(P^\bullet, \mathcal{P}^\bullet)$ (Corollary 4.2). It is worth emphasizing that both $(P^\bullet, \hat{\mathcal{P}}^\bullet)$ and $(P^\bullet, \mathcal{P}^\bullet)$ have the same topology and the same metric (1) defined on $P^\circ \subset P^\bullet$.

One can wonder whether the asymptotic smoothness (smoothness at the singularity) plays any role in the context of physical investigations. But, from the mathematical point of view, the asymptotic smoothness requirement for physical fields is well motivated since the smoothness is a key notion within the theory of d-spaces and the space-time with singularity is a d-space. Smooth objects define a d-space's properties. One can say that non-smooth objects are "outside" the d-space's theory. In a sense, "non-smoothness" is a symptom of the theory inconsistency.

If one tries to model physical reality with the help a d-space then every physical field has to be smooth. Therefore, the asymptotic non-smoothness of the considered physical fields for $\Delta \neq 2\pi(1-1/n)$, $n = 1, 2, \ldots$ is a serious defect which is non removable without modifications of the d-structure $\mathcal{P}^\bullet$. Thus, the consistency assumption of the theory of physical fields (scalar and electromagnetic) on the cosmic string space-time in the context of the d-spaces theory leads to the following deficit angle "quantization" condition: $\Delta = 2\pi(1-1/n)$, $n = 1, 2, \ldots$.

One can compare the above results with the conclusions obtained by Aliev and Gal'tsov (Sect. 6 or [2]). The scalar and electromagnetic conical bremsstrahlung occurs only in the case of asymptotically non smooth scalar and electromagnetic elementary solutions on $(P^\bullet, \mathcal{P}^\bullet)$. In other words, the radiative scalar or electromagnetic A-B effects vanish under the assumption of asymptotic smoothness of solutions. The disappearance of the conical bremsstrahlung was treated by Aliev and Gal'tsov as a "quantization" condition analogously to the well known effect for the quantum-mechanical A-B effect for a magnetic flux. The asymptotic smoothness assumption plays a similar role.

Acknowledgements I thank Prof. Michael Heller and Prof. Krzysztof Ruebenbauer for their comments, valuable discussion and suggestions.

Appendix A: Differential Spaces

The fundamental notions and theorems of the theory of differential spaces in the sense of Sikorski can be found in a monograph by R. Sikorski [23] or in [8, 9, 11, 21, 22]. Here I give the definitions and theorems necessary to follow the present paper. Informations about spaces more general than d-spaces in the sense of Sikorski can be found in [3, 4, 13, 14, 25–27].

Let $\mathcal{C}_0$ be a set of real functions on M.

Definition A.1 The set of functions

$$\mathrm{sc}(\mathcal{C}_0) := \{f = \omega(\varphi_1, \varphi_2, \dots, \varphi_n)\colon \varphi_1, \varphi_2, \dots, \varphi_n \in \mathcal{C}_0,\ \omega \in \mathrm{C}^\infty(\mathbb{R}^n),\ n \in \mathbb{N}\}$$

is said to be the closure with respect to superposition with smooth functions from $\mathrm{C}^\infty(\mathbb{R}^n)$ for every $n \in \mathbb{N}$.

A function $f\colon M \to \mathbb{R}$ is local $\mathcal{C}_0$-function if for every $p_0 \in M$ there is a neighbourhood $U \in \mathrm{top}(M)$ and $\varphi \in \mathcal{C}_0$ such that $f|_U = \varphi|_U$.

Definition A.2 The set of all local $\mathcal{C}_0$-functions on M denoted by $(\mathcal{C}_0)_M$ is called the closure with respect to localization.

Details can be found in [8, 23].

Definition A.3 The set $\mathcal{C} = \mathrm{Gen}(\mathcal{C}_0) := (\mathrm{sc}(\mathcal{C}_0))_M$ is said to be generated by $\mathcal{C}_0$. Then the $\mathcal{C}_0$ is called the set of generators.

Theorem A.1 *Let $\mathcal{C}_0$ be a set of real functions on M and $\mathcal{C}$ the set of functions generated by $\mathcal{C}_0$; $\mathcal{C} := \mathrm{Gen}(\mathcal{C}_0)$. Then*

1) $\mathrm{top}(\mathcal{C}) = \mathrm{top}(\mathcal{C}_0)$,
2) *$\mathcal{C}$ is a d-structure*,
3) $\mathcal{C}_0 \subset \mathcal{C}$,
4) *$\mathcal{C}$ is the smallest (in the sense of inclusion) d-structure containing $\mathcal{C}_0$.*

Proof can be found in [9]. □

Definition A.4 If the set of generators $\mathcal{C}_0$ of a d-structure $\mathcal{C}$ is finite then the resulting d-space $(M, \mathcal{C})$ is said to be finitely generated.

Let $(M, \mathcal{C})$ be a d-space. One can define the following equivalence relation

$$\forall p, q \in M : p\ \rho_H q \Leftrightarrow \forall \alpha \in \mathcal{C} : \alpha(p) = \alpha(q). \tag{7}$$

If an equivalence class (with respect to ρ_H) $[p] \neq \{p\}$ for $p \in M$, the topological space $(M, \mathrm{top}(\mathcal{C}))$ is not Hausdorff.

Let π_{ρ_H} and $\tilde{\pi}^*_{\rho_H}$ denote the following maps

$$\pi_{\rho_H}\colon M \to M/\rho_H, \qquad \pi_{\rho_H}(p) = [p],$$
$$\tilde{\pi}^*_{\rho_H}\colon \mathbb{R}^{M/\rho_H} \to \mathbb{R}^M, \qquad \tilde{\pi}^*_{\rho_H}(\alpha) = \alpha \circ \pi_{\rho_H},$$

where $\alpha \in \mathbb{R}^{M/\rho_H}$.

The set $\mathcal{C}/\rho_H := {\tilde{\pi}^*_{\rho_H}}^{-1}(\mathcal{C})$ of real functions on M/ρ_H forms a d-structure $\mathcal{C}/\rho_H$ on M/ρ_H. The d-structure is said to be coinduced d-structure from $\mathcal{C}$ by the mapping $\tilde{\pi}^*_{\rho_H}$ [31].

Thus, the quotient space M/ρ_H can be equipped with the d-structure $\mathcal{C}/\rho_H$ forming a Hausdorff d-space $(M/\rho_H, \mathcal{C}/\rho_H)$. It is easy to see that

$$\pi_{\rho_H}: M \to M/\rho_H$$

is a smooth mapping between $(M, \mathcal{C})$ and $(M/\rho_H, \mathcal{C}/\rho_H)$ and, in addition,

$$\pi^*_{\rho_H} := \tilde{\pi}^*_{\rho_H}\,|_{\mathcal{C}/\rho_H}, \qquad \pi^*_{\rho_H}: \mathcal{C}/\rho_H \to \mathcal{C}$$

is the isomorphism of algebras $\mathcal{C}/\rho_H$ and $\mathcal{C}$ [18]. This result is very useful in the following form

Corollary A.1 Φ *is a smooth function on* $(M/\rho_H, \mathcal{C}/\rho_H)$ *iff* $\tilde{\Phi} := \pi^*_{\rho_H}(\Phi)$ *is a smooth function on* $(M, \mathcal{C})$.

Let $(\tilde{P}, \tilde{\mathcal{P}})$ be a finitely generated d-space with the d-structure $\tilde{\mathcal{P}}$ generated by the set of functions $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n\ \}$; $\tilde{\mathcal{P}} = \mathrm{Gen}\{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n\}$, $\tilde{\alpha}_k: \tilde{P} \to \mathbb{R}, i = 1, 2, \dots, n$. Then $(\tilde{P}/\rho_H, \tilde{\mathcal{P}}/\rho_H)$ is also a finitely generated d-space with d-structure $\tilde{\mathcal{P}}/\rho_H = \mathrm{Gen}(\alpha_1, \alpha_2, \dots, \alpha_n)$ where $\alpha_i: \tilde{P}/\rho_H \to \mathbb{R}$, $i = 1, 2, \dots, n$ are given by the following formula $\pi^*_{\rho_H}(\alpha_i) = \tilde{\alpha}_i$. In other words $\alpha_i([p]) := \tilde{\alpha}_i(p)$, $p \in \tilde{P}$, $[p] \in \tilde{P}/\rho_H$.

Definition A.5 Let $\mathcal{B}$ be a set of real functions on M. A linear mapping $\mathbf{V}: \mathcal{C} \to \mathcal{B}$ such that

$$\mathbf{V}(\alpha\beta) = \mathbf{V}(\alpha)\beta + \alpha\mathbf{V}(\beta),$$

for any $\alpha, \beta \in \mathcal{C}$, is said to be a vector field on $(M, \mathcal{C})$. A vector field is smooth if $\mathcal{B} \subset \mathcal{C}$.

The set of all smooth vector fields on a d-space $(M, \mathcal{C})$ is a module over $\mathbb{R}$ and is denoted by $\mathbf{X}(M)$.

Let $\pi_{\rho_H}: M \to M/\rho_H$ and $\pi^*_{\rho_H}: \mathcal{C}/\rho_H \to \mathcal{C}$ be the mappings as above. Then the mapping

$$\pi^\#_{\rho_H}: \mathbf{X}(M) \to \mathbf{X}(M/\rho_H),$$
$$\pi^\#_{\rho_H}(\mathbf{V}) := \pi^{*\,-1}_{\rho_H} \circ \mathbf{V} \circ \pi^*_{\rho_H}, \quad \mathbf{V} \in \mathbf{X}(M)$$

is an isomorphism of modulae $\mathbf{X}(M)$ and $\mathbf{X}(M/\rho_H)$ [19]. Let us formulate this result in form useful in the present paper.

Corollary A.2 $\tilde{\mathbf{V}}$ *is a smooth vector field on* $(M, \mathcal{C})$ *iff* $\mathbf{V} := \pi^\#_{\rho_H}(\tilde{\mathbf{V}})$ *is smooth on* $(M/\rho_H, \mathcal{C}/\rho_H)$.

Definition A.6 Let $(M, \mathcal{C})$ be any d-space. A function $\beta: M \to \mathbb{R}$ is said to be differentially dependent on functions $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{C}$ at a point $p \in M$ if there exist a neighbourhood $U \in \mathrm{top}(\mathcal{C})$ of p and a function $\omega \in C^\infty(\mathbb{R}^n)$ such that

$$\beta|_U = \omega(\alpha_1, \alpha_2, \dots, \alpha_n)|_U.$$

Definition A.7 A set $\{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \mathcal{C}$ is said to be differentially independent (d-independent) at a point $p \in M$ if no function α_i, for $i \in \{1, 2, \dots, n\}$, depends differentially on the remaining functions at p.

Lemma A.1 *Let* $(M, \mathcal{C})$ *be a d-space with the d-structure* $\mathcal{C}$ *generated by the set of functions* $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. *The set of functions* $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *is d-independent at* $p \in M$ *iff* $\dim T_p M = n$.

Proof can be found in [16]. See also [8, 15]. □

Appendix B: Definitions and Formulae for the Cosmic String Space-Time

Let $(\tilde{P}^\circ, \tilde{\mathcal{P}}^\circ)$ be an auxiliary d-space, where $\tilde{P}^\circ := \mathbb{R} \times (0, \infty) \times \langle 0, 2\pi \rangle \times \mathbb{R}$ is a "parameter space", $\tilde{\mathcal{P}}^\circ := \mathrm{Gen}(\tilde{\alpha^\circ}_0, \tilde{\alpha^\circ}_1, \ldots, \tilde{\alpha^\circ}_4)$ and functions $\tilde{\alpha}_i^\circ : \tilde{P}^\circ \to \mathbb{R}$, $i = 0, 1, \ldots, 4$ are given by the following formulae

$$\tilde{\alpha}_0^\circ(\tilde{p}) := t,$$
$$\tilde{\alpha}_1^\circ(\tilde{p}) := \rho \cos\phi,$$
$$\tilde{\alpha}_2^\circ(\tilde{p}) := \rho \sin\phi,$$
$$\tilde{\alpha}_3^\circ(\tilde{p}) := z,$$
$$\tilde{\alpha}_4^\circ(\tilde{p}) := \rho,$$

where $\tilde{p} \in \tilde{P}^\circ$. $(\tilde{P}^\circ, \tilde{\mathcal{P}}^\circ)$ is not Hausdorff. The finitely generated d-space $(P^\circ, \mathcal{P}^\circ)$, $P^\circ = \tilde{P}^\circ/\rho_H$, $\mathcal{P}^\circ = \tilde{\mathcal{P}}^\circ/\rho_H := \mathrm{Gen}(\alpha_0^\circ, \alpha_1^\circ, \ldots, \alpha_4^\circ)$, $\alpha_i^\circ : P^\circ \to \mathbb{R}$, $\alpha_i^\circ([p]) := \tilde{\alpha}_i^\circ(p)$, $i = 0, 1, \ldots, 4$ (see Appendix A), is a Hausdorff topological space and the following lemma holds

Lemma B.1 *The d-space* $(P^\circ, \mathcal{P}^\circ)$ *is diffeomorphic to* $(C^\circ \times \mathbb{R}^2, (\mathcal{E}_5)_{C^\circ \times \mathbb{R}^2})$, *where* $\mathcal{E}_5 = C^\infty(\mathbb{R}^5)$.

Proof can be found in [8]. □

A similar lemma is valid for the cosmic string space-time with singularity. Let $(\tilde{P}^\bullet, \tilde{\mathcal{P}}^\bullet)$ be an auxiliary prolonged d-space, where $\tilde{P}^\bullet := \mathbb{R} \times \langle 0, \infty) \times \langle 0, 2\pi \rangle \times \mathbb{R}$, $\tilde{\mathcal{P}}^\bullet := \mathrm{Gen}(\tilde{\alpha^\bullet}_0, \tilde{\alpha^\bullet}_1, \ldots, \tilde{\alpha^\bullet}_4)$ and $\tilde{\alpha}_i^\bullet : \tilde{P}^\bullet \to \mathbb{R}$ are defined as follows

$$\tilde{\alpha}_i^\bullet(\tilde{p}_\bullet) := \lim_{\tilde{p} \to \tilde{p}_\bullet} \tilde{\alpha}_i^\circ(\tilde{p}),$$

where $\tilde{p} \in \tilde{P}^\circ$, $\tilde{p}_\bullet \in \tilde{P}^\bullet$ and $i = 0, 1, \ldots, 4$. $(\tilde{P}^\bullet, \tilde{\mathcal{P}}^\bullet)$ is also not Hausdorff.

Let $P^\bullet = \tilde{P}^\bullet/\rho_H$ and $\mathcal{P}^\bullet = \tilde{\mathcal{P}}^\bullet/\rho_H = \mathrm{Gen}(\alpha_0^\bullet, \alpha_1^\bullet, \ldots, \alpha_4^\bullet)$, $\alpha_i^\bullet([p]) = \tilde{\alpha}_i^\bullet$, $i = 0, 1, \ldots, 4$. The differential space $(P^\bullet, \mathcal{P}^\bullet)$ is a Hausdorff topological space.

Lemma B.2 *The d-space* $(P^\bullet, \mathcal{P}^\bullet)$ *is diffeomorphic to* $(C^\bullet \times \mathbb{R}^2, (\mathcal{E}_5)_{C^\bullet \times \mathbb{R}^2})$.

Proof can be found in [8]. □

References

1. Aliev, A.N.: Gravitational radiation in the field of supermassive cosmic string. Class. Quantum Gravity **10**, 2531–2538 (1993)

2. Aliev, A.N., Gal'tsov, D.V.: Gravitational Aharonov-Bohm radiation in string-generated conical space-time. Ann. Phys. **193**, 142–165 (1989)
3. Aronszajn, N.: Subcartesian and subriemannian spaces. Not. Am. Math. Soc. **14**, 111 (1967)
4. Chen, K.T.: Iterated path integrals. Bull. Am. Math. Soc. **83**, 831 (1977)
5. Ford, L.H., Vilenkin, A.: A gravitational analogue of the Aharonov-Bohm effect. J. Phys. A Math. Gen. **14**, 2353–2357 (1981)
6. Gott III, J.R.: Gravitational lensing effects of vacuum strings: exact solutions. Astrophys. J. **288**, 422 (1985)
7. Gruszczak, J.: Differential structure of space-time and physical fields. Acta Cosmol. **21**(2), 219–226 (1995). Proc. of conf.: Modern Methods in Classical and Quantum Gravity, Sintra (Portugal), 26–28 July 1995
8. Gruszczak, J., Heller, M.: Differential structure of space-time and its prolongations to singular boundaries. Int. J. Theor. Phys. **32**, 625–648 (1993)
9. Gruszczak, J., Heller, M., Multarzynski, P.: A generalization of manifolds as space-time models. J. Math. Phys. **29**, 2576–2580 (1988)
10. Gruszczak, J., Heller, M., Multarzynski, P.: Physics with and without the equivalence principle. Found. Phys. **19**, 607–618 (1989)
11. Gruszczak, J., Heller, M., Sasin, W.: Quasiregular singularity of a cosmic string. Acta Cosmol. **18**, 45–55 (1992)
12. Kay, B.S., Studer, U.M.: Boundary conditions for quantum mechanics on cones and fields around cosmic strings. Commun. Math. Phys. **139**, 103–139 (1991)
13. Marshall, Ch.D.: Calculus on subcartesian spaces. J. Differ. Geom. **10**, 551–574 (1975)
14. Mostov, M.A.: The differentiable space structures of Milnor classifying spaces-simplicial complex and geometric relations. J. Differ. Geom. **14**, 255–293 (1979)
15. Multarzynski, P., Sasin, W.: Algebraic characterization of the dimension of differential spaces. Suppl. Rend. Circ. Mat. Palermo II **22**, 193–199 (1989)
16. Multarzyński, P., Sasin, W.: On the dimension of differential spaces. Demonstr. Math. **23**, 405–415 (1990)
17. Penrose, R., Rindler, W.: Spinors and Space-Time, vol. 1. Cambridge University Press, Cambridge (1984). Chap. 4.
18. Sasin, W.: On equivalence relations on a differential space. Comment. Math. Univ. Carol. **29**, 529–539 (1988)
19. Sasin, W., Zekanowski, Z.: On locally finitely generated differential spaces. Demonstr. Math. **20**, 477–486 (1987)
20. Serebriany, E.M., Frolov, V.P., Skarzhinsky, V.D.: In: Proc. 4th Seminar on Quantum Gravity, Moscow. World Scientific, Singapore (1988)
21. Sikorski, R.: Abstract covariant derivative. Colloq. Math. **18**, 251–272 (1967)
22. Sikorski, R.: Differential modules. Colloq. Math. **24**, 45–79 (1971)
23. Sikorski, R.: Introduction to Differential Geometry. PWN (1972) (In Polish)
24. Smith, A.G.: Gravitational effects of an infinite straight cosmic string on classical and quantum fields. In: Gibbons, G.W., Hawking, S.W., Vachaspati, T. (eds.) The Formation and Evolution of Cosmic Strings, pp. 263–292. Cambridge University Press, Cambridge (1989)
25. Spallek, K.: Differenzierbare raume. Math. Ann. **180**, 269–296 (1969)
26. Spallek, K.: Rend. Math. 4 Ser.VI: 237 (1971a)
27. Spallek, K.: Rend. Math. 5 Ser.VI: 375 (1971b)
28. Staruszkiewicz, A.: Gravitation theory in three-dimensional space. Acta Phys. Pol. **24**, 735–740 (1963)
29. Vilenkin, A.: Phys. Rev. D **24**, 2082 (1981)
30. Vilenkin, A.: Phys. Rep. **121**, 263 (1985)
31. Waliszewski, W.: Regular and coregular mappings of differential spaces. Ann. Pol. Math. **30**, 263–281 (1975)